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LONG-RUN OPTIMAL BEHAVIOR IN A TWO-SECTOR ROBINSON–SOLOW–SRINIVASAN MODEL

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This paper studies the nature of long-run behavior in a two-sector model of optimal growth. Under some restrictions on the parameters of the model, we provide an explicit solution of the optimal policy function generated by the optimal growth model. Fixing the discount factor, we indicate how long-run optimal dynamics changes as a key technological parameter (labor output ratio) changes. For a particular configuration of parameter values, we also provide an explicit solution of the unique absolutely continuous invariant ergodic distribution generated by the optimal policy function.

Keywords: Bifurcation Analysis, Labor–Output Ratio, Lyapunov Exponent, Invariant Measure

1. INTRODUCTION

The purpose of this paper is to describe the nature of long-run behavior in a two-sector model of optimal growth, and to indicate how that behavior changes with respect to changes in a technological parameter of the model. The topic is best viewed as an exercise in trying to understand the relationship between a dynamic optimization model and the optimal policy function generated by it. A basic question of interest in this area is whether the exercise of dynamic optimization imposes some restrictions on the nature of the optimal policy function.

A number of studies[1] devoted to this question demonstrated that optimal programs can exhibit a variety of long-run behavior, including cycles and chaos, by constructing suitable examples in the context of various economic models, which can all be considered to be particular cases of a "reduced-form" model (Ω, u, ρ), where Ω is a transition possibility set determined by technological possibilities,

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u is a (reduced-form) utility function defined on this set, and ρ is the discount factor. An aspect of the constructed examples in this literature is that the utility function depends on the chosen discount factor. Thus, in a basic sense, this literature fails to address the question of how optimal behavior changes with respect to a parameter of the model when all other parameters of the model are held fixed (known as "sensitivity" or "bifurcation" analysis).

Progress on bifurcation analysis with respect to the *discount factor* has been made by several authors. In their seminal work, Benhabib and Nishimura (1985) provided an analysis of changes in the local stability behavior of the stationary optimal stock with changes in the discount factor. Boldrin and Deneckere (1990) and Nishimura and Yano (1995) studied specific classes of two-sector neoclassical models, which can generate optimal cycles and chaos, and showed how such optimal behavior is affected by changes in the discount factor. Long-run optimal behavior in a general reduced-form model (which allows for period-two optimal cycles but no more complicated behavior than that) was undertaken in Mitra and Nishimura (2001), where a complete bifurcation diagram was obtained without explicit solution for the optimal policy function.

The present investigation can be seen as a continuation of this line of research. It is clear from the literature that further progress on bifurcation analysis of long-run optimal behavior will be difficult unless one is able to solve explicitly for the optimal policy function, at least for substantial ranges of the parameters of the model. Our research on the Robinson–Solow–Srinivasan (RSS) model [see Khan and Mitra (2005) for references to the literature on which this model is based] has indicated that a two-sector version of it would provide an appropriate framework because it is both tractable (enabling explicit solution of the optimal policy function) and rich in the variety of optimal dynamics that it can generate (enabling a bifurcation analysis that would indicate how simple optimal dynamics gives way to more complex optimal behavior as the parameters of the model change).

The RSS two-sector model is specified by three parameters (a, d, ρ), where a is the labor–output ratio in the investment good sector, d is the depreciation factor of capital, and ρ is the discount factor. In this paper, in the context of this RSS model, we provide three main results.

First, we provide sufficient conditions on the parameters of the model (a, d, ρ) under which the optimal policy correspondence is an optimal policy function, which can be explicitly specified to be a "check map" (an upside-down asymmetric tent map), whose slopes are determined entirely by the technological parameters a and d.

Second, after transforming the check map into an asymmetric tent map, we show how a theorem of Lindstrom and Thunberg (2008) can be applied to our framework to provide a bifurcation analysis of the optimal dynamics generated by the model. Especially noteworthy in this regard is the result that, if we fix the depreciation factor, d, and the discount factor, ρ, and carry out a bifurcation analysis with respect to the *labor–output ratio*, a, we see that there is a threshold value $\tilde{a}$, such that (i) if the labor–output ratio a is higher than $\tilde{a}$, then there is a period-two optimal cycle, which attracts optimal paths from almost every initial stock, whereas (ii) if the labor–output ratio a is lower than $\tilde{a}$, then the dynamical system exhibits a positive Lyapunov exponent (so that optimal trajectories starting from very similar initial stocks tend to separate from each other exponentially over time, indicating a sensitive dependence on initial conditions that is a characteristic of chaotic behavior). The typical period-doubling route to chaos is noticeably absent.

Third, for a particular configuration of the parameters of the model, we show how a result of Boyarsky and Scarowsky (1979) can be applied to our model, to provide a completely rigorous derivation of an explicit solution for the absolutely continuous invariant ergodic distribution of the dynamical system generated by the optimal policy function of the model. The explicit solution makes it possible to predict precisely the fraction of time that a typical optimal trajectory will spend in any region of the state space.

2. THE TWO-SECTOR ROBINSON–SOLOW–SRINIVASAN MODEL

A single consumption good is produced by infinitely divisible labor and machines, with the further Leontief specification that a unit of labor and a unit of machines produce a unit of the consumption good. In the investment-goods sector, only labor is necessary to produce machines, with $a > 0$ units of labor producing a unit of machines. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbf{N}$, where $\mathbf{N}$ is the set of nonnegative integers. Thus, in the canonical formulation surveyed in McKenzie (1986), the collection of production plans (x, x'), the amount x' of machines in the next period (tomorrow) from the amount x available in the current period (today), is given by the *transition possibility set.* Here it takes the specific form $\Omega = \{(x, x') \in \mathbf{R}^2_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}$, where $z \equiv (x' - (1-d)x)$ is the amount of machines that are produced, and $z \geq 0$ and $az \leq 1$ respectively formalize constraints on the irreversiblity of investment and the use of labor. Associated with Ω is the transition correspondence, $\Gamma : \mathbf{R}_+ \to \mathbf{R}_+$, given by $\Gamma(x) = \{x' \in \mathbf{R}_+ : (x, x') \in \Omega\}$. For any $(x, x') \in \Omega$, one can also consider the amount y of machines available for the production of the consumption good, leading to a correspondence: $\Lambda : \Omega \longrightarrow \mathbf{R}_+$ with $\Lambda(x, x') = \{y \in \mathbf{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a(x' - (1-d)x)\}$.

Welfare is derived only from the consumption good and is represented by a linear function, normalized so that y units of the consumption good yield a welfare level y. A *reduced-form utility function* $u : \Omega \to \mathbf{R}_+$ with $u(x, x') = \max\{y \in \Lambda(x, x')\}$ indicates the maximum welfare level that can be obtained today if one starts with x machines today, and ends up with x' machines tomorrow, where $(x, x') \in \Omega$. Intertemporal preferences are represented by the present value of the stream of welfare levels, using a discount factor $\rho \in (0, 1)$.

A *two-sector RSS model* $\mathcal{G}$ consists of a triple (a, d, ρ), and the following concepts apply to it. A *program* from x_o is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_o$, and for all $t \in \mathbf{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) = \max \Lambda((x(t), x(t+1))$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$, and associated with it is a *gross investment sequence* $\{z(t+1)\}$, defined by $z(t+1) = (x(t+1) - (1-d)x(t))$ for all $t \in \mathbf{N}$. It is easy to check that every program $\{x(t), y(t)\}$ is bounded by $\max\{x(0), 1/ad\} \equiv M(x(0))$, and so $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \infty$. A program $\{\bar{x}(t), \bar{y}(t)\}$ from x_o is called *optimal* if $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \leq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$ holds for every program $\{x(t), y(t)\}$ from x_o. A program $\{x(t), y(t)\}$ is called *stationary* if for all $t \in \mathbf{N}$, we have $(x(t), y(t)) = (x(t+1), y(t+1))$. A *stationary optimal program* is a program that is stationary and optimal.

The parameter $\xi = (1/a) - (1-d)$ plays an important role in all of our subsequent analysis. It represents the marginal rate of transformation of capital today into that of tomorrow, given full employment of both factors. In what follows, and without further mention, we always assume that the parametrs (a, d) of the RSS model are such that $\xi > 1$. For more details, highlighting the geometric and analytical aspects of the model, the reader is referred to Khan and Mitra (2006, 2007b).

2.1. Dynamic Programming

Using standard methods, one can establish that there exists an optimal program from every $x \in X \equiv [0, \infty)$. Thus, we can define a *value function*, $V : X \to \mathbf{R}$, by

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)), \tag{1}$$

where $\{\bar{x}(t), \bar{y}(t)\}$ is an optimal program from x. Then, it is straightforward to check that V is concave, nondecreasing, and continuous on X. Further, it can be verified that V is, in fact, increasing on X.

It can be shown that for each $x \in X$, the Bellman equation

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\} \tag{2}$$

holds. For each $x \in X$, we denote by $h(x)$ the set of $x' \in \Gamma(x)$ that maximize $\{u(x, x') + \delta V(x')\}$ among all $x' \in \Gamma(x)$. That is, for each $x \in X$, we have $h(x) = \arg[\max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}]$.Thus, a program $\{x(t), y(t)\}$ from $x \in X$ is an optimal program from x if and only if it satisfies the equation $V(x(t)) = u(x(t), x(t+1)) + \delta V(x(t+1))$ for $t \geq 0$; that is, if and only if $x(t+1) \in h(x(t))$ for $t \geq 0$.

We call h the *optimal policy correspondence*. It can be shown to be upper hemicontinuous. When this correspondence is a function, we refer to it as the *optimal policy function* (OPF). Thus, when an OPF exists, it is necessarily continuous.

2.2. The Modified Golden Rule

A *modified golden rule* is a pair $(\hat{x}, \hat{p}) \in \mathbf{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$ and $u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x)$ for all $(x, x') \in \Omega$.

The existence of a modified golden rule has already been established in Khan and Mitra (2007b). We reproduce that result here (without proof) for ready reference. A distinctive feature of our model is that we can describe the modified golden rule stock explicitly in terms of the parameters of the model, and it is independent of the discount factor; the modified golden rule price, of course, depends on the discount factor.

PROPOSITION 1. *Define* $(\hat{x}, \hat{p}) = (1/(1 + ad),\ 1/(1 + \rho\xi))$. *Then* $(\hat{x}, \hat{x}) \in \Omega$, *where* $\hat{x}$ *is independent of* ρ, *and*

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega. \tag{3}$$

The connection between the value function in the dynamic programming approach and the modified golden rule may be noted as follows. Given a modified golden rule $(\hat{x}, \hat{p}) \in \mathbf{R}^2_+$, we know that $\hat{x}$ is a stationary optimal stock [see McKenzie (1986, p. 1305)]. Consequently, we have $V(\hat{x}) = \hat{x}/(1 - \rho)$.

2.3. Basic Properties of the Optimal Policy Correspondence

The basic properties of the optimal policy correspondence, with no additional restrictions on the parameters of our model, have already been described in Khan and Mitra (2007b). We summarize these properties hereafter. This helps us to present an explicit solution of the optimal policy correspondence in the next section.

To this end, we describe three regions of the state space: $A = [0, \hat{x}]$, $B = (\hat{x}, k)$, $C = [k, \infty)$, where $k = \hat{x}/(1 - d)$. We further subdivide the region B into two regions as follows: $D = (\hat{x}, 1)$, $E = [1, k)$. In addition, we define a function, $g : X \to X$, by

$$g(x) = \begin{cases} (1 - d)x & \text{for } x \in C \\ \hat{x} & \text{for } x \in B \\ (1/a) - \xi x & \text{for } x \in A \end{cases} \tag{4}$$

and a function $H : X \to X$ by

$$H(x) = \begin{cases} (1 - d)x & \text{for } x \in C \cup E \\ (1/a) - \xi x & \text{for } x \in A \cup D. \end{cases} \tag{5}$$

We refer to g as the "pan map," in view of the fact that its graph resembles a pan. We refer to H as the "check map," following the terminology of Day and Pianigiani (1991), because its graph resembles the standard check mark.

Finally, we define a correspondence, $G : B \to X$, by

$$G(x) = \begin{cases} [(1-d)x, \hat{x}] & \text{for } x \in E \\ [(1/a) - \xi x, \hat{x}] & \text{for } x \in D. \end{cases} \quad (6)$$

PROPOSITION 2. *The optimal policy correspondence, h, satisfies*

$$h(x) \subset \begin{cases} \{g(x)\} & \textit{for all } x \in A \cup C \\ G(x) & \textit{for all } x \in B. \end{cases} \quad (7)$$

Further, when $\rho\xi > 1$, the optimal policy correspondence, h, is an optimal policy function, given uniquely by the pan map, g. When $\rho\xi = 1$, the optimal policy correspondence, h, is given by the correspondence G for all $x \in B$.

It should be clear from this result that the only part of the optimal policy correspondence for which we do not have an explicit solution (in general) is for the middle region of stocks, given by $B = (\hat{x}, k) = D \cup E$, when $\rho\xi < 1$.

3. EXPLICIT SOLUTION OF THE OPTIMAL POLICY FUNCTION

In this section we present an explicit solution[2] of the optimal policy function under further technological and discount factor restrictions on the RSS model (a, d, ρ). Specifically, we provide three separate restrictions under which the optimal policy correspondence is a function, given uniquely by the check map, H.

The first result imposes no restriction on the technological parameters, so that any (a, d) permissible by the description of the model in Section 2 can be allowed. It imposes a strong restriction on the discount factor, demanding that it be less than the labor–output ratio, a.

PROPOSITION 3. *Suppose the RSS model (a, d, ρ) satisfies $\rho < a$. Then its optimal policy correspondence, h, is the function given by the check map H.*

Considerable interest attaches to the set of technological parameters (a, d) that satisfy the condition $H^2(1) < k$. It indicates that the second iterate of H of the unit initial stock is less than k. Because $\xi > 0$, the second iterate of H of the unit stock always exceeds the modified golden rule stock, $\hat{x}$. Thus, the condition ensures that $H^2(1)$ falls *inside* the interval $(\hat{x}, k) \equiv B$, the zone of stocks for which we do not have an explicit form of the optimal policy correspondence in general; we refer to this conveniently as the "inside case." If $H^2(1) > k$, then $H^2(1)$ falls *outside* the interval $(\hat{x}, k) \equiv B$, and we refer to this as the "outside case." If $H^2(1) = k$, then we refer to this as the "borderline case," it being the borderline between the inside and outside cases.

The three cases can be restated explicitly in terms of the technological parameters (a, d), or equivalently (ξ, d), of the model. In fact, one can verify that

$$H^2(1) \leq k \Longleftrightarrow [\xi - (1/\xi)]\,(1-d) \leq 1. \quad (8)$$

To see this, let us define the closed intervals $J_1 = [1-d, \hat{x}]$; $J_2 = [\hat{x}, 1]$; $J_3 = [1, k]$, and denote $H^2(1)$ by k'. Denote the length of the interval J_2 by θ. Notice that H maps J_2 onto J_1, and the relevant slope for this domain is $(-\xi)$, so that the length of J_1 is $\xi\theta$. Further, H maps J_3 onto J_1, and the relevant slope for this domain is $(1-d)$, so that the length of $J_3 = \xi\theta/(1-d)$. Thus, the length of $J_2 \cup J_3 = [\hat{x}, k]$ is $\{\theta + [\xi\theta/(1-d)]\}$. On the other hand, H maps J_1 onto $[\hat{x}, k']$, and the relevant slope for this domain is $(-\xi)$, so that $k' > \hat{x}$ and $[k' - \hat{x}] = \xi^2\theta$. Thus, we obtain

$$k' \leq k \Longleftrightarrow \xi^2 \leq 1 + [\xi/(1-d)] \tag{9}$$

One can rewrite the right-hand inequality in (9) as

$$\xi \leq \{(1/\xi) + [1/(1-d)]\}. \tag{10}$$

By transposing terms, (10) is the same as the right-hand side of (8). Thus, using the equivalence in (9), we have demonstrated that the equivalence in (8) holds.

When the technological parameters are restricted so that (ξ, d) satisfies the inequality on the right-hand side of (8) (that is, we are in the inside or the borderline case), then a weaker restriction than $\rho < a$ on the discount factor ρ ensures that the optimal policy correspondence is a function given by the check map, H.

PROPOSITION 4. *Suppose the RSS model* (a, d, ρ) *satisfies* $[\xi - (1/\xi)](1 - d) \leq 1$, *and*

$$\rho < \sqrt{a/\xi} \equiv \eta. \tag{11}$$

Then its optimal policy correspondence, h, is the function given by the check map H.

Note that because $\xi = (1/a) - (1-d)$, we have $\xi < (1/a)$, and so $(a/\xi) > a^2$. Thus $\eta > a$, and because (11) requires only that $\rho < \eta$, it ensures that the optimal policy correspondence, h, is the check map H not only when $\rho < a$, but also when $\rho \in [a, \eta)$. The price one pays for the weaker discount factor restriction is of course that Proposition 4 applies under the technological restriction $[\xi - (1/\xi)](1-d) \leq 1$, whereas Proposition 3 holds without any such restriction.

If one strengthens the technological restriction even further so that (ξ, d) satisfies the inequality $\xi(1-d) \leq 1$, then an even weaker restriction than (11) on the discount factor ρ ensures that the optimal policy correspondence is a function given by the check map H.

PROPOSITION 5. *Suppose the RSS model* (a, d, ρ) *satisfies* $\xi(1-d) \leq 1$, *and*

$$\rho < 1/\xi. \tag{12}$$

Then its optimal policy correspondence, h, is the function given by the check map, H.

Note that because $\xi = (1/a) - (1-d)$, we have $a\xi < 1$, so that $(a/\xi) < (1/\xi^2)$ and $\eta < (1/\xi)$. Because (12) requires only that $\rho < (1/\xi)$, it ensures that the optimal policy correspondence h is the check map H not only when $\rho < \eta$, but also when $\rho \in [\eta, (1/\xi))$. The trade-off for the even weaker discount factor restriction (12) is that Proposition 5 applies under the technological restriction $\xi(1-d) \leq 1$, whereas Proposition 4 holds under the technological restriction $[\xi - (1/\xi)](1-d) \leq 1$, which is clearly weaker.

The economic interpretation of the sufficient conditions enruring that the optimal policy function is given by the check map can be seen as follows. The sufficient condition $\rho < a$ relates the intertemporal rate of substitution between consumption today and tomorrow, which is ρ (because the utility function is linear in consumption), to a *lower bound* on the intertemporal rate of transformation, given by a. Thus, if $x \in (1, k)$ and $x' > (1-d)x$, then one can increase consumption a little (by say $\varepsilon > 0$) by increasing labor input in the consumption goods sector by ε, and reducing labor input in the investment good sector by ε. This will reduce the capital stock in the next period by (ε/a). One can then reduce consumption in the next period by (ε/a) by reducing labor input in the consumption good sector in the next period. It can be seen that the economy will have a higher capital stock at the end of this two-period variation. Thus, if $\rho < a$, then this variation will increase the discounted sum of utilities by $[\varepsilon - \rho(\varepsilon/a)] > 0$, and leave the economy with a higher capital stock at the end of this two-period variation. Thus, if $x \in (1, k)$, it is *not optimal* to have $x' > (1-d)x$, and therefore the optimal policy must have $x' = (1-d)x$. If $x \in (\hat{x}, 1]$, the situation is a bit more complicated, because the previously described variation made use of the fact that the capital stock x exceeds the consumption level y. (This is always true when $x \in (1, k)$, because $x > 1 \geq y$, but is clearly not true when $x \in (\hat{x}, 1]$). However, if $(\hat{x}, 1]$, and $x' > (1/a) - \xi x$, then $x' > (1-d)x$ is automatically satisfied, and further $y < x$, so that the preceding argument can be used to show that it is not optimal to have $x' > (1/a) - \xi x$. Thus the optimal policy must have $x' = (1/a) - \xi x$ when $x \in (\hat{x}, 1]$. This establishes that the check map is optimal.

The sufficient conditions in Propositions 4 and 5 can be seen as refinements of the basic argument. When $\rho > a$, the two-period variation does not yield a dominating action. However, at the end of the two-period variation, the economy has a higher capital stock by a magnitude equal to $\xi(\varepsilon/a)$, and one has to evaluate the discounted value of future streams of extra consumption (beyond the two periods) that can be obtained from this higher capital stock. This consideration leads to sufficient conditions that involve $\rho < \eta$ (in Proposition 4) and $\rho < (1/\xi)$ (in Proposition 5), which can hold even when $\rho > a$.

4. BIFURCATION ANALYSIS

In this section, we provide a bifurcation analysis of the long-run optimal dynamics generated by the RSS model. The analysis can be naturally divided into two parts. First, we need a full bifurcation analysis of the dynamics of the check map, which is described purely by the technological parameters (a, d). Second, we need to graft onto that analysis technological and discount factor restrictions (described in the previous section) under which the check map represents the optimal policy function, so that the bifurcation analysis of the dynamics of the check map describes changes in long-run *optimal* behavior with respect to the parameters of the model.

4.1. Dynamics of the Check Map

We first describe how the long-run dynamics of the check map depend on the technological parameters (a, d) [equivalently the parameters (ξ, d)]. This is accomplished by (i) transforming the check map through a linear transformation of the variable (capital stock) to an *asymmetric tent map*, and (ii) applying the rather complete analysis of the dynamics of the asymmetric tent map presented by Lindström and Thunberg (2008).

When we specialize this analysis to the "inside case," a rather remarkable picture emerges regarding the optimal dynamics. Generically, either one has $\xi(1-d) < 1$, in which case there is a period-two cycle that attracts trajectories from almost all initial conditions; or one has $\xi(1-d) < 1$, in which case, *every* periodic point is *repelling*, and almost all trajectories have *positive Lyapunov exponents*. Thus, trajectories starting from very similar initial stocks tend to separate from each other exponentially over time, indicating a strong form of sensitive dependence on initial conditions, which is a characteristic of chaotic behavior.[3]

To paraphrase, one has very simple dynamics for $\xi(1-d) < 1$, and immediately after crossing the bifurcation point to the range $\xi(1-d) > 1$, one has chaotic behavior. The usual period-doubling route to chaos, found in the symmetric tent map, is noticeably absent.

To proceed, let us define $L = (1/ad)$ and $Y = [0, L]$. Then Y is a natural state space for the variable x (the capital stock), because if $x \in Y$, then $H(x) \in Y$, so that Y is an invariant set. If $x > L$, then some finite iterate of H belongs to Y, so no aspect of long-run dynamics is lost by confining attention to Y as the state space. To simplify notation, we denote by H again the restriction of the check map to Y; the dynamical system can then be denoted by (Y, H), where H is given by

$$H(x) = \begin{cases} (1/a) - \xi x, & \text{for } x \in [0, 1] \\ (1-d)x, & \text{for } x \in (1, L]. \end{cases} \tag{13}$$

Given that the check map is like a mirror image of an asymmetric tent map, the route to follow is to convert the map H given by (13) to a map F, which is an asymmetric tent map, by a linear transformation of the variable x.

If we define the linear transformation $Z = L - x$, the dynamical system (Y, H) is transformed into the dynamical system (Y, F), where Z is the state variable, and F is given by the asymmetric tent map

$$F(Z) = \begin{cases} P - \xi(Z - Z^*), & \text{for } Z \in [Z^*, L] \\ P + (1-d)(Z - Z^*), & \text{for } Z \in [0, Z^*), \end{cases} \tag{14}$$

where $Z^* = L - 1$ and $P = (1/a) + (1-d)Z^*$.[4]

The dynamical system (Y, F) has been studied in detail by Lindström and Thunberg (2008), who conclude from their study that the system exhibits "a very special type of dynamical pattern with respect to the bifurcation parameters" and the "transition to chaos is far from standard." It is possible to study the asymmetric tent map directly, but because we would like to apply the results of Lindström and Thunberg (2008), we follow them to make one more linear transformation of variables to reduce the number of parameters from three to two, so that the key parameters become the slopes of the two straight lines of the tent map F.

If we define the linear transformation $z = [(Z - Z^*)/d]$, the dynamical system (Y, F) is transformed into the dynamical system (Q, f), where z is the state variable, f is given by the asymmetric tent map

$$f(z) = \begin{cases} 1 - \xi z, & \text{for } z \in [0, (1/d)] \\ 1 + (1 - d)z, & \text{for } z \in [-(L - 1)/d, 0), \end{cases} \qquad \textbf{(15)}$$

and $Q \equiv [-(L - 1)/d, 1/d]$ is the state space.[5]

In Theorem 4.1 of their paper, Lindström and Thunberg provide a complete description (with proofs) of the dynamics of (15) for all (generic) values of the slopes of the two branches of the asymmetric tent map. Our parameters ξ and $(1-d)$ correspond to their parameters k and c, respectively. However, in our case, there are several restrictions on the parameters; we have $0 < (1 - d) < 1$ and $\xi > 1$. So only some of the cases of their Theorem 4.1 apply in our case.[6] Further, it will facilitate our discussion of the RSS model to have a statement of their theorem in terms of the original variable (capital stock, x) of the RSS model.[7]

THEOREM 6. *Trajectories generated by the dynamical system (Y, H) have the following properties:*

(I) If $\xi > 1$ and $\xi(1 - d) < 1$, then there exists a period-two cycle attracting trajectories from almost all initial conditions $x \in Y$.

(II) If $\xi > 1$ and $\xi(1 - d) > 1$, and there is $n \geq 3$, such that

$$\frac{1 - (1 - d)^{n-1}}{d(1 - d)^{n-2}} = \left[\sum_{s=0}^{n-2} \frac{1}{(1 - d)^s}\right] < \xi < \frac{1}{(1 - d)^{n-1}}, \qquad \textbf{(16)}$$

then there is a period-n cycle attracting trajectories from almost all initial conditions $x \in Y$.

(III) If $\xi > 1$ and $\xi(1 - d) > 1$, and ξ is outside the regions specified by (16), trajectories from almost all initial conditions $x \in Y$ enter and remain in the interval $[(1 - d), (1 - d) + d\xi]$ and have positive Lyapunov exponent.

Let us note that $q(n) \equiv \sum_{s=0}^{n-2}[1/(1 - d)^s]$ appearing in (16) is monotone increasing in n, and so $q(n)$ attains a minimum at $n = 3$ among all $n \in \mathbf{N}$ with $n \geq 3$. Thus, in order for (16) to apply, it is *necessary* to have

$$\xi > q(3) = [(2 - d)/(1 - d)]. \qquad \textbf{(17)}$$

In the “inside” and “ borderline” cases, we have $H^2(1) \leq k$, which is equivalent to the condition $[\xi - (1/\xi)](1-d) \leq 1$. Now, because $\xi > 1$, this condition implies that

$$\xi(1-d) \leq 1 + [(1-d)/\xi] < (2-d). \tag{18}$$

Clearly, (18) implies that (17) cannot hold. That is, in the “inside” and “ borderline” cases of the RSS model, (16) *never* holds, and the only possibilities are (I) or (III) in Theorem 1. We state this as a separate result.

PROPOSITION 7. *Consider the RSS model with* (a, d) *satisfying the restriction* $[\xi - (1/\xi)](1-d) \leq 1$. *Then trajectories generated by the dynamical system* (Y, H) *have the following properties:*

(i) If $\xi(1-d) < 1$, *then there exists a period-two cycle attracting trajectories from almost all initial conditions* $x \in Y$.

(ii) If $\xi(1-d) > 1$, *trajectories from almost all initial conditions* $x \in Y$ *enter and remain in the interval* $[(1-d), (1-d) + d\xi]$ *and have positive Lyapunov exponent.*

4.2. Bifurcation Analysis with Respect to the Labor–Output Ratio

Propositions 3, 4, and 5 can be combined with Theorem 1 to provide a variety of results relating to bifurcation analysis of the *optimal* dynamics generated by the RSS model. We will confine ourselves to one such application, which will illustrate the general approach. This will be to the “inside” or “borderline” cases, because that will allow us to apply the striking threshold result contained in Proposition 7.

To this end, define the following functions of d on the open interval (0, 1) :

$$\left.\begin{array}{l} \sigma(d) = 1/(1-d);\ \xi(d) = (\tfrac{1}{2})[\sigma(d) + \sqrt{\sigma(d)^2 + 4}], \\ a(d) = 1/[\xi(d) + (1-d)];\ \eta(d) = 1/\{\sqrt{\xi(d)[\xi(d) + (1-d)]}\}. \end{array}\right\} \tag{19}$$

That is, given $d \in (0, 1)$, $\xi(d)$ is chosen to satisfy $\{\xi(d) - [1/\xi(d)]\}(1-d) = 1$. Then $a(d)$ is chosen to satisfy $\xi(d) = (1/a(d)) - (1-d)$, and $\eta(d)$ is chosen to satisfy $\eta(d) = \sqrt{a(d)/\xi(d)}$.

Now pick any $\hat{d} \in (0, 1)$ and any $\hat{\rho} \in (0, \eta(\hat{d}))$ and fix them. Define

$$a^* = 1/[2 - \hat{d}];\ \ \bar{a} = a(\hat{d}). \tag{20}$$

Then, for every $a \in [\bar{a}, a^*)$, the RSS model $(a, \hat{d}, \hat{\rho})$ satisfies

$$\xi(\hat{d}) \geq \xi > 1 \text{ and } [\xi - (1/\xi)](1 - \hat{d}) \leq 1, \tag{21}$$

where $\xi \equiv (1/a) - (1 - \hat{d})$. Further, by (20) and (21),

$$\sqrt{(a/\xi)} \geq \sqrt{a(\hat{d})/\xi(\hat{d})} = \eta(\hat{d}) > \hat{\rho}. \tag{22}$$

Thus, by Proposition 5, the optimal policy correspondence h of the RSS model $(a, \hat{d}, \hat{\rho})$ is a function given by the check map H. This allows us to conduct a bifurcation analysis of long-run optimal dynamics for the RSS model $(a, \hat{d}, \hat{\rho})$ with respect to the labor–output ratio a, as a varies in the interval $[\bar{a}, a^*)$. We state our finding formally as a result.

PROPOSITION 8. *Consider an RSS model for which the depreciation factor is fixed at some* $\hat{d} \in (0, 1)$, *and the discount factor is fixed at* $\hat{\rho} \in (0, \eta(\hat{d}))$. *Define* a^* *and* $\bar{a}$ *as in (30),* $\Theta \equiv [\bar{a}, a^*)$, *and*

$$\tilde{a} \equiv 1/[\sigma(\hat{d}) + (1 - \hat{d})]. \tag{23}$$

(i) If $a \in \Theta$ *and* $a > \tilde{a}$, *then there exists a period-two optimal cycle attracting optimal trajectories from almost all initial conditions* $x \in Y$.

(ii) If $a \in \Theta$ *and* $a < \tilde{a}$, *optimal trajectories from almost all initial conditions* $x \in Y$ *enter and remain in the interval* $[(1 - \hat{d}), (1 - \hat{d}) + \hat{d}\xi]$ *and have positive Lyapunov exponent.*

Proposition 7 indicates that there is a *threshold* labor–output ratio, $\tilde{a}$. If the actual labor–output ratio is higher than this threshold level, the typical optimal dynamics will be very simple, with almost all optimal trajectories converging to a (necessarily unique) period-two optimal cycle. If the actual labor–output ratio is lower than this threshold level, the typical optimal dynamics will be complicated.

5. EXPLICIT SOLUTION OF AN INVARIANT DENSITY

The previous section has demonstrated ranges of parameter values of the RSS model for which the dynamical system will have a positive Lyapunov exponent; one can also show that the dynamical system will have an invariant distribution that is absolutely continuous with respect to Lebesgue measure on the reals.[8] The nature of the invariant distribution of the dynamical system is, however, far from clear in general.

Our objective in this section is to demonstrate how one can explicitly solve for a *unique ergodic* absolutely continuous invariant measure for a dynamical system generated by the optimal policy function of the RSS model, for a particular parameter configuration resulting in the "borderline case." Although we focus on a particular case, our method will exhibit the following features:

(i) We will show how the general method[9] of Boyarsky and Scarowsky (1979) allows one to compute the unique invariant density by calculating the eigenvector of a Markov transition matrix, A, defined by the check map H.

(ii) In solving for the invariant density, we will see that the parametric restriction is equivalent to stating that the matrix $(I - A)$ is singular.

(iii) We will be able to infer the "sojourn time" of a "typical optimal trajectory" in any subinterval of $[(1 - d), k]$, which is the support of the unique invariant density, by invoking the ergodic theorem. The explicit solution shows that in the borderline case, the capital stock along a typical optimal trajectory will spend half the time above the golden rule stock.

Consider the RSS model (a, d, ρ) in the borderline case; that is, (a, d) are such that

$$[\xi - (1/\xi)](1 - d) = 1. \tag{24}$$

Further, ρ is such that

$$\rho < \sqrt{a/\xi} \equiv \eta. \tag{25}$$

Then, by Proposition 4, the optimal policy correspondence h is a function, given by the check map H.

5.1. A Markov Matrix and Its Eigenvector

To apply the method of Boyarsky and Scarowsky (1979) to obtain the invariant density associated with the optimal policy function H, we have to verify that H belongs to the class of functions to which their results apply. To this end, we proceed as follows.

Let us define the closed intervals $J_1 = [1 - d, \hat{x}]$, $J_2 = [\hat{x}, 1]$, $J_3 = [1, k]$, and the corresponding open intervals $I_1 = [1 - d, \hat{x}]$, $I_2 = [\hat{x}, 1]$, $I_3 = [1, k]$.

Define $J = J_1 \cup J_2 \cup J_3$, and consider the map $H : J \to \mathbf{R}$. Then H is in fact a map from J to J. Consider the set of open intervals $\Im = \{I_1, I_2, I_3\}$. Then for each I_r (with $r \in \{1, 2, 3\}$), H is C^2 on I_r, and can be extended to a C^2 function on J_r. We refer to $\Im$ as a *partition* of J, and the set of points $M = \{(1 - d), \hat{x}, 1, k\}$ as the *partition points*, following Boyarsky and Scarowsky (1979).

Except for the turning point $x = 1$, H is differentiable on J, and $H'(x) \geq (1 - d) > 0$ for all $x \in J \backslash \{1\}$. There exist $b \in I_1$ and $b' > k$ such that $H(b) = H(b') = 1$. Thus, H is not differentiable at $H(b) = H(b') = 1$, and so the function H^2 is not differentiable at b and at b'. By the chain rule, H^2 is differentiable at all points in J except the points in $N = \{1, b\}$. Further, we have $dH^2(x)/dx \geq \xi(1 - d)$ on $J \backslash N$. Using (24), we have $\xi(1 - d) > 1$. This verifies that H^2 is *expanding*.

Because $H(k) = \hat{x}$, $H(\hat{x}) = \hat{x}$, $H(1) = (1 - d)$, and $H((1 - d)) = k$, the function H maps the partition points into the partition points.

Finally, note that $H(I_1) \supset I_2 \cup I_3$, $H(I_2) \supset I_1$ and $H(I_3) \supset I_1$. Thus, for each I_r, I_s, where $r, s \in \{1, 2, 3\}$, there exist positive integers m and n such that $H^m(I_r) \supset I_s$ and $H^n(I_s) \supset I_r$. Thus, H satisfies the *communication property* of Boyarsky and Scarowsky (1979).

We have now verified that $H : J \to J$ belongs to the class $\mathcal{C}$ as defined in Boyarsky and Scarowsky (1979, p. 244). Thus, by their Theorem 1 (p. 246), there is a unique absolutely continuous invariant measure. It is ergodic. Further, by their Theorem 3 (p. 259), the density corresponding to this measure is *piecewise constant*.[10]

The density can be explicitly calculated[11] by defining a Markov transition matrix $A = (a_{ji})$ by

$$a_{ji} = \begin{cases} |H_i'|^{-1} & \text{if } I_j \subset H_i(I_i) \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

where H_i is the restriction of H to I_i. Then the solution to the invariant density is given by $E = (e_1, e_2, e_3)$, where $AE = E$. That is, E is the eigenvector of A corresponding to the eigenvalue of 1. Denoting by ϕ the invariant density, we have: $\phi(x) = e_i$ for $x \in I_i$ for $i \in \{1, 2, 3\}$, and $\phi(x) = 0$ for $x \in \mathbf{R}\backslash J$.

Using (26), we can obtain the matrix A in our case as follows:

$$A = \begin{bmatrix} 0 & \xi^{-1} & (1-d)^{-1} \\ \xi^{-1} & 0 & 0 \\ \xi^{-1} & 0 & 0 \end{bmatrix}. \tag{27}$$

With A given by (27), $\det(A - I)$ can be evaluated as follows:

$$\det(A - I) = [1/\xi^2] - [1 - \{1/\xi(1-d)\}] = 0, \tag{28}$$

where the second equality in (28) follows from the parametric restriction (24). Thus the parametric restriction (24) ensures that there is a nontrivial solution to $AE = E$. This solution can be found by noting that

$$\begin{aligned} &\text{(i) } \xi^{-1}e_2 + (1-d)^{-1}e_3 = e_1, \\ &\text{(ii) } \xi^{-1}e_1 = e_2, \\ &\text{(iii) } \xi^{-1}e_1 = e_3 \end{aligned} \tag{29}$$

must hold at any solution E to $AE = E$, by using (27).

Clearly, $e_2 = e_3$ from (29)(ii) and (29)(iii). Denoting this common value by e, we get from (29)(i)

$$[\xi^{-1} + (1-d)^{-1}]e = e_1. \tag{30}$$

Thus, E is solved except for a normalization. Because E is the density, which is piecewise constant on the intervals of the partition, we have

$$\begin{aligned} \nu(I_1) &= \xi\theta e_1 = \xi\theta[\xi^{-1} + (1-d)^{-1}]e, \\ \nu(I_2) &= \theta e, \\ \nu(I_3) &= [\xi\theta/(1-d)]e, \end{aligned} \tag{31}$$

where ν is the invariant measure corresponding to the invariant density, and $\theta = (1-\hat{x})$ is the length of J_2. Thus, e can be found by setting $\nu(I_1)+\nu(I_2)+\nu(I_3) = 1$, and then E can be found to be

$$[e_1\ e_2\ e_3] = [(\xi+1)/2\xi d \quad (\xi+1)/2\xi^2 d \quad (\xi+1)/2\xi^2 d]. \tag{32}$$

For a numerical example, consider $a = (3/8), d = (1/3)$. Then we have $\theta = (1-\hat{x}) = (1/9)$, $\xi = 2$, and $[\xi - (1/\xi)](1-d) = 1$, so the parametric restriction (24) is satisfied, Then, using (32), the density ϕ is given by $\phi(x) = (9/4)$ for $x \in I_1$, and $\phi(x) = (9/8)$ for $x \in I_2 \cup I_3$.

5.2. Sojourn Time

Because ν is ergodic, the Birkhoff pointwise ergodic theorem applies. Because $f(x) > 0$ for Lebesgue almost every x, the measure ν is equivalent to Lebesgue measure on J; that is, if S is a (Lebesgue) measurable set in J, then $\nu(S) > 0$ if and only if $m(S) > 0$, where m is Lebesgue measure on the reals. Thus, we can infer that for Lebesgue almost every $x \in J$, we have

$$\lim_{T\to\infty}(1/T)\sum_{t=0}^{T-1}\chi[I_r; H^t(x)] = \int_J \chi(I_r; x) f(x)dx = \int_{I_r} f(x)dx = \nu(I_r), \tag{33}$$

where $\chi(I_r; \cdot)$ is the characteristic function of the interval I_r.

The left-hand side expression in (33) is the "sojourn time" of the optimal trajectory, starting from x, in the interval I_r. That is, it is the fraction of time spent by the optimal trajectory, starting from x, in the interval I_r. The formula (33) tells us that this can be measured by $\nu(I_r)$ for Lebesgue almost every $x \in J$. Thus, $\nu(I_r)$ measures the fraction of the time spent by the "typical" optimal trajectory in the interval I_r.

Note from (31) and (32) that $\nu(I_1) = \xi[(1/\xi) + \{1/(1-d)\}]\theta e = [1 + \{\xi/(1-d)\}]\theta e = \nu(I_2) + \nu(I_3)$. Thus, $\nu(I_1) = 1 - \nu(I_1)$ and $\nu(I_1) = (1/2)$, and $\nu(I_2 \cup I_3) = (1/2)$. So the " typical" optimal trajectory spends half the time above the golden rule stock and half the time below it.

NOTES

1. See, for example, Boldrin and Montrucchio (1986), Deneckere and Pelikan (1986), Majumdar and Mitra (1994) and Nishimura et al. (1994).
2. The proofs of Propositions 3,4, and 5 in Section 3 are omitted. Proofs of Propositions 3 and 4 can be found in Khan and Mitra (2007a). Proofs of Proposition 5 can be found in Khan and Mitra (2006, 2010).
3. A positive Lyapunov exponent is one of the standard indicators of a chaotic dynamical system. For a definition and discussion of the concept, see for example Eckman and Ruelle (1985).
4. The map given in (14) is the standard form of the *threshold autoregressive model* (TAR(1) model) studied in nonlinear time series analysis by Tong (1990).
5. The map F corresponds to the map $T(\cdot)$ described in equation (1) of Lindström and Thunberg (2008). Note that $(P - Z^*) = d > 0$, so the map f corresponds to the map $t(\cdot)$ described in equation (2) of Lindström and Thunberg (2008). Because they analyze the map $t(\cdot)$ in detail in their paper, we are now all set to directly apply their results.
6. Specifically, parts (I), (II), and (III) of our Theorem 6 correspond to parts (f), (g), and (h), respectively, of Theorem 4.1 in Lindström and Thunberg (2008).
7. For the RSS model under consideration, the "almost all initial conditions" phrase in Theorem 6 can be replaced by "all stocks except the golden-rule stock" in result (I). A geometric demonstration of this is contained in Khan and Mitra (2010).
8. This can be done by applying the result of Lasota and Yorke (1973).
9. After our paper had been completed, an interesting paper of Matsumoto (2005) came to our attention. Matsumoto uses the Boyarsky–Scarowsky method to obtain the invariant density for a piecewise linear map along the lines that we follow in Section 5.1. Our principal interest in obtaining the invariant density explicitly is that it allows us to relate it to the parameters of our optimal growth

model, and thereby make a prediction of the sojourn time of a typical optimal trajectory above the modified golden rule. Because Matsumoto is not concerned with an optimal growth model, this motivation as well as the application is absent in his study.

10. Day and Pianigiani (1991) provide an explicit solution of an invariant measure for the dynamical system associated with their model by assuming that it is piecewise constant.

11. See Boyarsky and Scarowsky (1979, p. 260) for a complete explanation.

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